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Supertwistors, massive superparticles and κ-symmetry

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Abstract: We consider a $D = 4$ two-twistor lagrangian for a massive particle that incorporates the mass-shell condition in an algebraic way, and extend it to a two-supertwistor model with $N = 2$ supersymmetry and central charge identified with the mass. In the purely supertwistorial picture the two $D = 4$ supertwistors are coupled through a Wess-Zumino term in their fermionic sector. We demonstrate how the κ-gauge symmetry appears in the purely supertwistorial formulation and reduces by half the fermionic degrees of freedom of the two supertwistors; a formulation of the model in terms of κ-invariant degrees of freedom is also obtained. We show that the κ-invariant supertwistor coordinates can be obtained by dimensional (D=6$\rightarrow$$D$=4) reduction from a $D = 6$ supertwistor. We derive as well by 6$\rightarrow$4 reduction the $N = 2$, D=4 massive superparticle model with Wess-Zumino term introduced in 1982. Finally, we comment on general superparticle models constructed with more than two supertwistors.

Keywords: p-branes, Extended Supersymmetry, Superspaces.

JHEP01(2009)041

Contents

1. Introduction

The conformal structure of twistor theory (see e.g. [1]–[4]) implies that relativistic particles described by single twistors are massless [1, 3, 5, 6]. To describe massive particles at least two twistors are needed [7, 8, 3, 9]–[13] (*cf.* [14]). Indeed, with two $D = 4$ twistors

$$Z_{Ai} = (\lambda_{\alpha i}\,,\ \bar{\omega}_i^{\dot{\alpha}})\,, \quad i = 1,2\,, \quad \alpha, \dot{\alpha} = 1,2\,, \tag{1.1}$$

the standard formula for the composite fourmomentum[1]

$$p_{\alpha\dot{\beta}} = \lambda_{\alpha i}\,\bar{\lambda}_{\dot{\beta} i} \quad , \quad p_\mu = \frac{1}{\sqrt{2}} p_{\alpha\dot{\beta}} (\sigma_\mu)^{\dot{\beta}\alpha} \quad , \tag{1.2}$$

implies the algebraic relation

$$p_\mu p^\mu = p_{\alpha\dot{\beta}}\, p^{\dot{\beta}\alpha} = |M|^2 \quad , \tag{1.3}$$

where the Lorentz-invariant bilinear[2]

$$M = \frac{1}{\sqrt{2}} \lambda_{\alpha i}\,\lambda^{\alpha i} \quad \left(\bar{M} = \frac{1}{\sqrt{2}} \bar{\lambda}_{\dot{\alpha} i}\,\bar{\lambda}^{\dot{\alpha} i} \right) \tag{1.4}$$

[1] $\lambda^{\alpha i} = \epsilon^{ij} \lambda^{\alpha}_{\ j}$; $\epsilon^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\epsilon_{ij}\epsilon^{jk} = \delta_j^k$. In general, $a_{\alpha\dot{\beta}} \equiv \frac{1}{\sqrt{2}} \sigma^\mu_{\alpha\dot{\beta}} a_\mu$, $\sigma_{\alpha\dot{\beta}} = (\sigma^0, \sigma^i)$, $\sigma^{\dot{\beta}\alpha} = (\sigma^0, -\sigma^i)$, $i = 1,2,3$. Then, $a^\nu = \frac{1}{\sqrt{2}} a_{\alpha\dot{\beta}} \sigma^{\nu\ \dot{\beta}\alpha}$, $a_{\alpha\dot{\beta}} a^{\dot{\beta}\gamma} = \frac{1}{2} \delta^\alpha_\gamma\, a^\mu a_\mu$ and $a_{\alpha\dot{\beta}} b^{\dot{\beta}\alpha} = a \cdot b$, $a^2 = (a^0)^2 - \vec{a}^2$. Complex conjugated spinors are denoted by $(\lambda_{\alpha i})^* \equiv \bar{\lambda}_{\dot{\alpha} i}$, etc.

[2] In [11, 12] the variable $f = \lambda_{\alpha 1} \lambda^{\alpha}_{\ 2} = \frac{1}{\sqrt{2}} M$ was used in place of M.

gives the composite complexified mass and breaks the conformal invariance down to the Poincaré one. To obtain a real mass m it suffices to consider a twistor theory invariant under the phase transformations

$$\lambda'_{\alpha i} = e^{i\varphi}\,\lambda_{\alpha i} \quad , \quad \bar{\lambda}'_{\dot{\alpha} i} = e^{-i\varphi}\,\bar{\lambda}_{\dot{\alpha} i} \quad ; \tag{1.5}$$

by a suitable gauge fixing, a real $M \to |M| = m$ is obtained.

Twistorial particle models constructed from several twistors are known [3, 7−9]; in particular a two-twistor model was proposed recently to describe free relativistic particles with mass and spin [11]–[13]. These considerations confirm that the introduction of a non-vanishing Pauli-Lubański vector, describing the relativistic spin fourvector in terms of twistor coordinates, requires at least two twistors [9]. In this paper we extend the two-twistor particle dynamics by considering two $N = 2$ supertwistors to describe the degrees of freedom of our elementary system. The supersymmetric two-twistor geometry will be arranged in a way that leads algebraically to the condition

$$\Xi = M \,, \tag{1.6}$$

where Ξ is the complex central charge of the D=4, N=2 superPoincaré algebra. We shall use the phase symmetry (1.5) so that the value M of the central charge Ξ in (1.6) becomes a real parameter. An interesting outcome of our approach will be the description of the fermionic κ-gauge transformations in a purely twistorial formulation of the massive superparticle model. Further, it may be shown that the presence of the mass in the twistorial framework reduces the conformal supersymmetry realized in terms of the two N=2 supertwistors to the N=2 superPoincaré symmetry with a composite central charge given by (1.4).

The plan of the paper is the following: In section 2 we introduce a massive two-twistor particle model without supersymmetry using a hybrid spinor/spacetime formulation. We also present there two other equivalent spacetime and purely twistorial geometry formulations and exhibit their relation with the associated six-dimensional massless particle model. In section 3 we introduce the $N = 2$ supersymmetric extension of the bosonic massive model, with degrees of freedom described by two ($i = 1, 2$) $N = 2$ ($r = 1, 2$) supertwistors [15]

$$\mathcal{Z}_{Ri} = (Z_{Ai}, \eta_{ri}) = (\lambda_{\alpha i}, \bar{\omega}^{\dot{\alpha}}{}_i, \eta_{ri}) \,, \tag{1.7}$$

where Z_{Ai} and η_{ri} are, respectively, Grassmann even and odd. This supersymmetric model will be formulated first in a hybrid spinor/superspace geometric framework [5]. We discuss (*cf.* [16, 17]) the constraints and, in particular, the fermionic first class constraints that generate the κ-transformations with odd gauge parameters [18, 19].

Further, we present in section 3 a purely supertwistorial (two-supertwistor) formulation of our model. Its lagrangian will be the sum of two terms,

(i) a first one describing the free two-supertwistor lagrangian, and

(ii) an additional bilinear term in the fermionic sector which couples the two supertwistors; this will turn out to be a supertwistorial Wess-Zumino (WZ) term.

JHEP01(2009)041

The model is completed by adding the Lagrange multipliers that describe its algebraic constraints.

Section 4 is devoted to quantizing the fermionic sector of the two supertwistor model, to uncover the quaternionic structure behind it and to calculating the fermionic gauge κ-transformations. These κ-symmetries reduce by half (from four to two) the number of complex Grassmannian supertwistorial degrees of freedom. We find new features of the two-supertwistor formulation with respect to single supertwistor models: the appearance of a WZ term and the presence of the fermionic κ-gauge transformations associated with the non-physical fermionic degrees of freedom in the multi-supertwistorial formulation. If we impose the quaternionic SU(2) Majorana condition in the fermionic sector, the redundant degrees of freedom of the two supertwistor coordinates described in the fermionic sector by the κ-transformations can be eliminated. Subsequently, we show also in section 4 that it is possible to introduce κ-invariant fermionic variables which describe the fermionic sector of our model in terms of two (rather than four) complex odd degrees of freedom. It is then seen that these κ-invariant fermionic variables can be interpreted as the dimensionally reduced ($D = 6 \rightarrow D = 4$) odd components of a six-dimensional complex supertwistor, satisfying SU(2)-Majorana conditions.

Section 5 exploits the above six-dimensional interpretation of the two-supertwistor model, using a pair of $N = 1$, $D = 4$ (super)twistors to describe a single $D = 6$ (super)twistor. We perform the $D = 6 \rightarrow D = 4$ dimensional reduction by taking the six-momentum coordinates p_4, p_5 as constants. As a result, we recover the D=4 N=2 massive superparticle model formulated by two of the present authors a quarter century ago, the first one with mass in the fermionic sector introduced by means of a WZ term [18]. The last section provides an outlook. In particular, it is argued there that using N D=4 supertwistors we can extend our construction and obtain a N-supertwistorial model with N supersymmetries as well as with $\frac{N(N-1)}{2}$ complex composite mass-like parameters playing the role of central charges. In particular, the $D = 4, N = 4$ model would be especially interesting, since its κ-invariant formulation could be given by the coordinates of a single $D = 10$ supertwistor with octonionic structure.

2. Massive particle model from $D = 6$ and its $D = 4$ twistorial picture

It is known that $D = 4$ massive free particle models can be obtained by dimensional reduction from massless particle ones in $D > 4$. Because the twistorial mass (1.4) is complex, we shall take $D = 6$ and denote $M = p_4 - ip_5$, $\bar{M} = p_4 + ip_5$ ((p_μ, p_4, p_5) are the six-momentum coordinates) and, similarly, $z = x^4 + ix^5$, $\bar{z} = x^4 - ix^5$. In the first order formalism, the Lagrangian of a D=6 massless particle can be written as

$$\mathcal{L}^B = p_\mu \dot{x}^\mu + \frac{1}{2}(M\dot{z} + \bar{M}\dot{\bar{z}}) + \frac{e}{2}(p^2 - |M|^2) \ , \ \mu = 0, 1, 2, 3 \quad . \tag{2.1}$$

Using eqs. (1.2) and (1.4) the 'mixed' or spinor/spacetime formulation of (2.1) is obtained, in which

$$\mathcal{L}^B = \bar{\lambda}_{\dot{\beta}i}\, \dot{x}^{\dot{\beta}\alpha}\, \lambda_{\alpha i} + \frac{1}{2\sqrt{2}} \left(\lambda_{\alpha i}\, \lambda^{\alpha i} \dot{z} + \bar{\lambda}_{\dot{\alpha} i}\, \bar{\lambda}^{\dot{\alpha} i} \dot{\bar{z}}\right) \ , \tag{2.2}$$

where the spacetime vector is expressed as a second rank spinor, $x_\mu = \frac{1}{\sqrt{2}}(\sigma_\mu)^{\dot\beta\alpha}x_{\alpha\dot\beta}$ (see the first footnote) and the D=6 zero mass shell condition, $p^k p_k = p^\mu p_\mu - |M|^2 = 0$ ($k = 0, 1, \ldots, 5$), is omitted because it is an identity in terms of the spinor variables $\lambda_{\alpha i}, \bar\lambda_{\dot\alpha i}$. Clearly, the lagrangian (2.2) is invariant under the rigid phase transformations (1.5) if the z's have a (*cf.* (1.5)) double U(1) charge,

$$z' = e^{-2i\varphi} z \qquad , \qquad \bar z' = e^{2i\varphi} \bar z \quad . \tag{2.3}$$

To obtain a purely twistorial description of the lagrangian (2.2) we now introduce a new Weyl spinor and its conjugate (ω_i^α, $\bar\omega_i^{\dot\alpha}$) and postulate for the components of the twistors $Z_{Ai} = (\lambda_{\alpha i}, \bar\omega_i^{\dot\alpha})$ the following incidence relations:

$$\begin{aligned} \omega^\alpha_{\ i} &= i\Big(\bar\lambda_{\dot\beta i}\, x^{\dot\beta\alpha} \;+ \frac{1}{\sqrt{2}}\lambda^{\alpha i}\, z\Big) , \\ \bar\omega^{\dot\alpha}_{\ i} &= -i\Big(x^{\dot\alpha\beta}\,\lambda_{\beta i} + \frac{1}{\sqrt{2}}\,\bar\lambda^{\dot\alpha i}\,\bar z\Big) , \end{aligned} \tag{2.4}$$

$(x^{\dot\alpha\beta})^\dagger = x^{\dot\beta\alpha}$ (x^μ real). Eqs. (2.4) generalize the well known Penrose formula [1]–[4]. Using them, the lagrangian (2.2) can be written as

$$\begin{aligned} \mathcal{L}^B &= -\frac{i}{2}\left(\dot\omega^\alpha_{\ i}\lambda_{\alpha i} + \dot{\bar\lambda}_{\dot\alpha i}\bar\omega_i^{\dot\alpha}\right) + \frac{i}{2}\left(\omega^\alpha_{\ i}\,\dot\lambda_{\alpha i} + \bar\lambda_{\dot\alpha i}\,\dot{\bar\omega}^{\dot\alpha}_{\ i}\right) \\ &= \frac{i}{2}\left(\bar Z^A_{\ i}\dot Z_{Ai} - \dot{\bar Z}^A_{\ i} Z_{Ai}\right) \quad , \end{aligned} \tag{2.5}$$

where the scalar product of two twistors is given by

$$\bar Z^A_{\ i}\, Z_{Ak} = \omega^\alpha_{\ i}\,\lambda_{\alpha k} \;+\; \bar\lambda_{\dot\alpha i}\bar\omega^{\dot\alpha}_{\ k} , \tag{2.6}$$

with $\bar Z_i^A = (\omega^\alpha{}_i, \bar\lambda_{\dot\alpha i})$.

One can check, using eqs. (2.4) and the relations

$$\lambda_{\alpha i}\,\lambda^\alpha_{\ j} = -\frac{1}{\sqrt{2}}\,\epsilon_{ij}\,M \quad , \qquad \bar\lambda_{\dot\alpha i}\,\bar\lambda^{\dot\alpha}_{\ j} = -\frac{1}{\sqrt{2}}\,\epsilon_{ij}\,\bar M \quad , \tag{2.7}$$

that

$$\bar Z^A_{\ i}\, Z_{Aj} = \frac{i}{2}\left(Mz - \bar M\bar z\right)\delta_{ij} \; . \tag{2.8}$$

Since $\bar Z^A_{\ k}\, Z_{Ak} = i\left(Mz - \bar M\bar z\right)$, it follows that the $z, \bar z$ coordinates may be removed by the relations (2.8), which can be rewritten as (see also [6])

$$\bar Z^A_{\ i}\, Z_{Aj} - \frac{1}{2}\,\delta_{ij}\,\bar Z^A_{\ k}\, Z_{Ak} \;= 0 , \tag{2.9}$$

an expression that does not fix the conformal norm of the twistors Z_{Ak} but states that the two norms are equal.

Summarizing, $\mathcal{L}^B$ in eq. (2.5) appears as the sum of two free twistor lagrangians, associated with two non-null, orthogonal twistors having the same non-vanishing length for $M \neq 0$. To specify that the purely twistorial lagrangian (2.5) describes a massive particle one has to incorporate relations (2.9) and (1.4) by means of suitable Lagrange multipliers.

3. Two-supertwistor model with $D = 4, N = 2$ supersymmetry and mass

a) Supersymmetrization of the model (2.2) and κ-transformations. The 'hybrid' or spinor/spacetime model (2.2) is supersymmetrized by replacing the translation-invariant differentials $dx^{\alpha\dot\beta}$, dz and $d\bar z$ by the corresponding supertranslation-invariant ones on $D = 4$, $N = 2$ superspace extenden by a central charge and parametrized by $(x_{\alpha\dot\beta}, z, \bar z;\ \theta_{\alpha r}, \bar\theta_{\dot\alpha r}\ )$, $r = 1, 2$. This is achieved by the replacements[3]

$$\dot x^{\dot\beta\alpha} \to \omega^{\dot\beta\alpha} = \dot x^{\dot\beta\alpha} - i\sqrt{2}\left(\dot\theta^{\alpha}_{\ r}\,\bar\theta^{\dot\beta}_{\ r} - \theta^{\alpha}_{\ r}\,\dot{\bar\theta}^{\dot\beta}_{\ r}\right)\ , \tag{3.1}$$

$$\dot z \to \omega = \dot z + 2i\,\theta_{\alpha r}\dot\theta^{\alpha r}\quad ,\quad \dot{\bar z} \to \bar\omega = \dot{\bar z} + 2i\,\bar\theta_{\dot\alpha r}\,\dot{\bar\theta}^{\dot\alpha r}\quad , \tag{3.2}$$

where $a_r b^r \equiv a_r \epsilon^{rs}\, b_s$ and $z, \bar z$ play the rôle of the D=4, N=2 superspace complex central charge coordinates. These replacements in (2.2) give the supersymmetric lagrangian

$$\mathcal{L}^{\rm SUSY} = \bar\lambda_{\dot\beta i}\,\omega^{\dot\beta\alpha}\,\lambda_{\alpha i} + \frac{1}{2\sqrt{2}}\left(\lambda_{\alpha i}\,\lambda^{\alpha i}\,\omega + \bar\lambda_{\dot\alpha i}\,\bar\lambda^{\dot\alpha i}\,\bar\omega\right)\ . \tag{3.3}$$

The action obtained from (3.3) is invariant under the supertranslations of the $N = 2$ superPoincaré group extended by a complex central charge,

$$\begin{aligned}
&x'^{\dot\beta\alpha} = x^{\dot\beta\alpha} - i\sqrt{2}(\epsilon^{\alpha}_{\ r}\,\bar\theta^{\dot\beta}_{\ r} - \theta^{\alpha}_{\ r}\,\bar\epsilon^{\dot\beta}_{r})\ , && \\
&\theta'_{\alpha r} = \theta_{\alpha r} + \epsilon_{\alpha r}\quad , && \bar\theta'_{\dot\alpha r} = \bar\theta_{\dot\alpha r} + \bar\epsilon_{\dot\alpha r}\ , \\
&z' = z + 2i\theta_{\alpha r}\,\epsilon^{\alpha r}\quad , && \bar z' = \bar z + 2i\,\bar\theta_{\dot\alpha r}\,\bar\epsilon^{\dot\alpha r}\ , \\
&\lambda'_{\alpha i} = \lambda_{\alpha i}\quad , && \bar\lambda'_{\dot\alpha i} = \bar\lambda_{\dot\alpha i}\quad ,
\end{aligned} \tag{3.4}$$

which leave $\lambda_{\alpha i}$ invariant. Expression (3.1) is also invariant under the U(2) internal transformation of the odd superspace coordinates $\theta_{\alpha r}, \theta_{\dot\alpha r}$; this symmetry is broken by the ω's of eq. (3.2) down to the $\mathrm{U}(2) \cap \mathrm{Sp}(2, C) = U\mathrm{Sp}(2) \approx \mathrm{SU}(2)$ internal symmetry.

The lagrangian (3.3) describes a superparticle in a mixed spinorial/superspace configuration space $\mathcal{M}^{(6;8|8)}$ parametrized by

$$\mathcal{M}^{(6;8|8)} = \{q^{\mathcal{M}}\} = (x^{\dot\alpha\beta}, z, \bar z; \lambda^{\alpha i}, \bar\lambda^{\dot\alpha i}|\theta^{\alpha}_{r}, \bar\theta^{\dot\alpha}_{r})\quad , \tag{3.5}$$

with 4+2+8 =14 real bosonic and 8 real fermionic coordinates. The canonical momenta $(\pi_{\alpha r} = \partial L/\partial\dot\theta^{\alpha}_{r}$, etc.)

$$\mathcal{P}_{\mathcal{M}} = \frac{\partial L}{\partial\dot q^{\mathcal{M}}} \equiv (p_{\dot\alpha\beta}, q, \bar q; \rho_{\alpha i}, \bar\rho_{\dot\alpha i}\,|\pi_{\alpha r}, \bar\pi_{\dot\alpha r})\ ,\ \mathcal{M} = 1\ldots 22\quad , \tag{3.6}$$

[3] The $\sqrt{2}$ in (3.1) is needed because in this way $\omega^\mu(\tau) = \frac{1}{\sqrt{2}}\sigma^{\mu}_{\ \alpha\dot\beta}\omega^{\dot\beta\alpha}(\tau) = \dot x^\mu - i(\dot\theta^\alpha_r\sigma^{\mu}_{\ \alpha\dot\beta}\bar\theta^{\dot\beta}_r - \theta^\alpha_r\sigma^{\mu}_{\ \alpha\dot\beta}\dot{\bar\theta}^{\dot\beta}_r)$ is the pull-back to to the worldline of the particle of the superspace Maurer-Cartan one-form $\Pi^\mu = dx^\mu - i(d\theta^\alpha_r\sigma^{\mu}_{\ \alpha\dot\beta}\bar\theta^{\dot\beta}_r - \theta^\alpha_r\sigma^{\mu}_{\ \alpha\dot\beta}d\bar\theta^{\dot\beta}_r)$, which is invariant under the $D = 4, N = 2$ superPoincaré transformations in eqs. (3.4).

JHEP01(2009)041

define the following set of primary constraints:

$$
\begin{aligned}
R_{\alpha\dot\beta} &:= p_{\alpha\dot\beta} - \lambda_{\alpha i}\,\bar\lambda_{\dot\beta i} = 0 \quad , \\
R &:= q - \frac{1}{2\sqrt{2}}\lambda_{\alpha i}\,\lambda^{\alpha i} = q - \frac{M}{2} = 0 \quad , \\
\bar R &:= \bar q - \frac{1}{2\sqrt{2}}\,\bar\lambda_{\alpha i}\,\bar\lambda^{\alpha i} = q - \frac{\bar M}{2} = 0 \quad , \\
R_{\alpha i} &:= \rho_{\alpha i} = 0 \quad , \\
R_{\dot\alpha i} &:= \bar\rho_{\dot\alpha i} = 0 \quad ,
\end{aligned}
\tag{3.7}
$$

and ($\bar G_{\dot\alpha r}{=}-(G_{\alpha r})^{+}$, $\bar\pi_{\dot\alpha r}{=}-(\pi_{\alpha r})^{+}$)

$$
G_{\alpha r} := \pi_{\alpha r} + i\sqrt{2}p_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}{}_{r} - iM\,\epsilon_{rs}\,\theta_{\alpha s} = 0 \quad , \tag{3.8}
$$

$$
\bar G_{\dot\alpha r} := \bar\pi_{\dot\alpha r} + i\sqrt{2}\theta^{\beta}{}_{r}p_{\beta\dot\alpha} - i\bar M\,\epsilon_{rs}\,\bar\theta_{\dot\alpha s} = 0 \quad . \tag{3.9}
$$

Let us restrict ourselves to the set (3.8), (3.9) of fermionic constraints, which determine the elements of the Poisson brackets (PB) matrix

$$
\mathcal{C}_{AB} = \begin{pmatrix} \{G_{\alpha r}, G_{\beta s}\},\ \{G_{\alpha r}, \bar G_{\dot\beta s}\} \\ \{\bar G_{\dot\alpha r}, G_{\beta s}\},\ \{\bar G_{\dot\alpha r}, \bar G_{\dot\beta s}\} \end{pmatrix} \quad .
$$

Using the canonical PB

$$
\{\theta_{\alpha r}\,, \pi_{\beta s}\} = \epsilon_{\alpha\beta}\,\delta_{rs} \quad , \quad \{\bar\theta_{\dot\alpha r}\,, \bar\pi_{\dot\beta s}\} = \epsilon_{\dot\epsilon\dot\beta}\delta_{rs} \quad , \tag{3.10}
$$

it follows that the four 4×4 blocks of the $\mathcal{C}_{AB}$ matrix are given by ($p_{\dot\beta\alpha} = (p_{\alpha\dot\beta})^{T}$)

$$
\mathcal{C}_{AB} = 2i\begin{pmatrix} -\epsilon_{\alpha\beta}\,\epsilon_{rs}M & \sqrt{2}\delta_{rs}\,p_{\alpha\dot\beta} \\ \sqrt{2}\delta_{rs}\,p_{\beta\dot\alpha} & -\epsilon_{\dot\alpha\dot\beta}\,\epsilon_{rs}\bar M \end{pmatrix} \quad .
$$

Using the formula for the determinant of a 2×2 blocks matrix,

$$
\det\mathcal{C} = \det A\cdot\det(D - CA^{-1}B) \quad , \quad \mathcal{C} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,, \tag{3.11}
$$

one finds that

$$
\det\mathcal{C} = 2^{8}(p^{2} - |M|^{2})^{4} \quad . \tag{3.12}
$$

The first constraint in (3.7) reproduces eq. (1.2), implying

$$
p^{2} - |M|^{2} = 0 \tag{3.13}
$$

(eq. (1.3)), and thus we conclude from (3.12) that the 8×8 PB constraints matrix (3.11) is of rank four.

We may now derive four first class fermionic constraints by multiplying respectively (3.8) by $p^{\alpha\dot\gamma}$ and (3.9) by $\epsilon_{rs}\, M$. This gives

$$\bar C_{\dot\beta r} = \pi^\alpha_r\, p_{\alpha\dot\beta} \; + \frac{M}{\sqrt2}\epsilon_{rs}\,\bar\pi_{\dot\beta s} - \frac{i}{\sqrt2}\bar\theta_{\dot\beta r}(p^2-|M|^2) = 0\,. \tag{3.14}$$

Equation (3.14) determines in principle four complex constraints, but their complex-conjugate ones are equivalent to them. Indeed, if we multiply the constraints ($C^+_{\beta r} = -(\bar C_{\dot\beta r})$)

$$C_{\beta r} = p_{\beta\dot\alpha}\,\bar\pi^{\dot\alpha}_r + \frac{\bar M}{\sqrt2}\epsilon_{rs}\,\pi_{\beta s} - \frac{i}{\sqrt2}\theta_{\beta r}(p^2-|M|^2) = 0\;, \tag{3.15}$$

by $p^{\dot\gamma\beta}$ we get back the constraints (3.14), plus terms that contain the factor $(p^2-|M|^2)$. Therefore our model has effectively only four real first class constraints, which in the 8-dimensional real Grassmann odd sector of the configuration space (3.5) generate four real odd gauge transformations. These are the κ-symmetries of the model (3.3) that eliminate the unphysical fermionic gauge degrees of freedom i.e., half of the odd $N{=}2$ superspace coordinates. The explicit expression of these κ-transformations, parametrized by a pair of anticommuting Weyl spinors $\kappa_{\alpha r}$ and their complex conjugates $\bar\kappa_{\dot\alpha r}$, is given by the graded Poisson brackets

$$\begin{aligned}
\delta_\kappa\,\theta^\alpha_r &:= \{\kappa^\beta_s C_{\beta s},\theta^\alpha_r\} = \kappa^\beta_s\{C_{\beta s},\theta^\alpha_r\} \;= -\epsilon_{rs}\frac{\bar M}{\sqrt2}\,\kappa^\alpha_s\;,\\
\delta_{\bar\kappa}\theta_{\alpha r} &:= \{\bar\kappa^{\dot\beta}_s\bar C_{\dot\beta s},\theta_{\alpha r}\} = \bar\kappa^{\dot\beta}_s\{\bar C_{\dot\beta s},\theta_{\alpha r}\} = -p_{\alpha\dot\beta}\bar\kappa^{\dot\beta}_r\quad,\\
\delta_\kappa\bar\theta^{\dot\alpha}_r &:= \{\kappa^\beta_s C_{\beta s},\bar\theta^{\dot\alpha}_r\} \;=\; \kappa^\beta_s\{C_{\beta s},\bar\theta^{\dot\alpha}_r\} \;=\; p^{\dot\alpha\beta}\,\kappa_{\beta r}\;,\\
\delta_{\bar\kappa}\,\bar\theta^{\dot\alpha}_r &:= \{\bar\kappa^{\dot\beta}_s\bar C_{\dot\beta s},\bar\theta^{\dot\alpha}_r\} = \bar\kappa^{\dot\beta}_s\{\bar C_{\dot\beta s},\bar\theta^{\dot\alpha}_r\} \;= -\epsilon_{rs}\frac{M}{\sqrt2}\,\bar\kappa^{\dot\alpha}_s\quad.
\end{aligned} \tag{3.16}$$

If δ_κ now denotes the variation under both κ and $\bar\kappa$, the variation of the four-dimensional spinor $(\theta_{\alpha r},\bar\theta^{\dot\alpha}_r)$ is written as

$$\delta_\kappa\begin{pmatrix}\theta_{\alpha r}\\ \bar\theta^{\dot\alpha}_r\end{pmatrix} = \begin{pmatrix}-\epsilon_{rs}\delta^\beta_\alpha\frac{\bar M}{\sqrt2} & -\delta_{rs}p_{\alpha\dot\beta}\\ \delta_{rs}p^{\dot\alpha\beta} & -\epsilon_{rs}\delta^{\dot\alpha}_{\dot\beta}\frac{M}{\sqrt2}\end{pmatrix}\begin{pmatrix}\kappa_{\beta s}\\ \bar\kappa^{\dot\beta}_s\end{pmatrix}\;. \tag{3.17}$$

The above matrix can be rewritten as the product

$$\begin{pmatrix}-\epsilon_{rt}\delta^\gamma_\alpha\frac{\bar M}{\sqrt2} & 0\\ 0 & -\epsilon_{rt}\delta^{\dot\alpha}_{\dot\gamma}\frac{M}{\sqrt2}\end{pmatrix}\begin{pmatrix}\delta_{ts}\delta^\beta_\gamma & -\epsilon_{ts}\frac{\sqrt2}{M}p_{\gamma\dot\beta}\\ \epsilon_{ts}\frac{\sqrt2}{\bar M}p^{\dot\gamma\beta} & \delta_{ts}\delta^{\dot\gamma}_{\dot\beta}\end{pmatrix}\;. \tag{3.18}$$

The first matrix just produces a scaling of the κ-transformations, and the second matrix is the sum of the four-dimensional unit matrix plus one with only non-zero 2×2 antidiagonal boxes that squares to one i.e., it is a projection operator. Thus, $\delta_\kappa\theta_r$ in eq. (3.17) has the standard projector structure effectively halving the parameters of the κ-transformations.

JHEP01(2009)041

The behaviour of the remaining configuration space variables (3.5) under δ_κ is given by

$$\begin{aligned}\delta_\kappa x^{\dot\beta\alpha} &= i\sqrt{2}(\delta_\kappa\theta^\alpha_{\ r}\,\bar\theta^{\dot\beta}_{\ r} - \theta^\alpha_r\,\delta_\kappa\bar\theta^{\dot\beta}_{\ r})\,,\\ \delta_\kappa z &= -2i\theta_{\alpha r}\,\delta_\kappa\,\theta^{\alpha r}\,,\\ \delta_\kappa \bar z &= -2i\bar\theta_{\dot\alpha r}\,\delta_\kappa\,\bar\theta^{\dot\alpha r}\,,\\ \delta_\kappa\lambda_{\alpha i} &= \delta_\kappa\,\bar\lambda_{\dot\alpha i} = 0 \quad .\end{aligned} \tag{3.19}$$

These relations differ from eqs. (3.4) by the replacement $\epsilon \to -\delta_\kappa\theta$; the relative minus sign characterizes κ-symmetry as a 'right' (local) supersymmetry (see e.g. [20]). The κ-transformations (3.16), (3.19), may be used to check explicitly the κ-invariance of the action based on (3.3).

b) From the hybrid (spinor/spacetime) formulation to the purely super twistorial one. To introduce a purely supertwistorial formulation of the model (3.3), eqs. (2.4) are further extended in the two supertwistors case by[4]

$$\begin{aligned}\omega^\alpha_i &= i\bar\lambda_{\dot\beta i}(x^{\dot\beta\alpha} - i\sqrt{2}\theta^\alpha_{\ r}\,\bar\theta^{\dot\beta}_{\ r}) + \frac{i}{\sqrt{2}}\,\epsilon^{ij}(\lambda^\alpha_{\ j}\,z + 2i\lambda_{\beta j}\,\theta^{\alpha r}\theta^\beta_r)\,,\\ \bar\omega^{\dot\alpha}_i &= -i(x^{\dot\alpha\beta} + i\sqrt{2}\theta^\beta_{\ r}\,\bar\theta^{\dot\alpha}_{\ r})\lambda_{\beta i} - \frac{i}{\sqrt{2}}\,\epsilon^{ij}(\bar\lambda^{\dot\alpha}_{\ j}\,\bar z - 2i\bar\lambda_{\dot\beta j}\,\bar\theta^{\dot\alpha}_{\ r}\bar\theta^{\dot\beta r})\,,\end{aligned} \tag{3.20}$$

which are the generalized incidence relations for the bosonic components of the Z_{Ai} part of $\mathcal{Z}_{Ri}$ (eq. (1.7)). These relations, which involve the fermionic superspace coordinates besides the real spacetime $x^{\dot\alpha\beta}$ and complex z variables (*cf.* eq. (2.4)), have to be complemented by those affecting the odd composite variables η_{ri}, $\bar\eta_{ri}$ that make up [15] the coordinates triple of the two D=4 $N=2$ supertwistors,

$$\bar{\mathcal{Z}}^R_{ri} = (\omega^\alpha_i\,,\,\bar\lambda_{\dot\alpha i}\,,\bar\eta_{ri}) \quad , \quad \mathcal{Z}_{Rri} = (\lambda_{\alpha i}\,,\,\bar\omega^{\dot\alpha}_i\,,\,\eta_{ri}) \quad , \quad i = 1,2 \quad , \tag{3.21}$$

where $r = 1,2$ is the N=2 supertwistor index. Eqs. (3.20) are accordingly supplemented by [15]

$$\eta_{ri} = \sqrt{2}\theta^\alpha_{\ r}\,\lambda_{\alpha i} \qquad , \qquad \bar\eta_{ri} = \sqrt{2}\bar\theta^{\dot\alpha}_{\ r}\,\bar\lambda_{\dot\alpha i} \quad . \tag{3.22}$$

Using (1.4), the above expressions can be inverted with the result

$$\theta^\alpha_{\ r} = \frac{\lambda^{\alpha j}\,\eta_{rj}}{M} \quad , \quad \bar\theta^{\dot\alpha}_{\ r} = \frac{\bar\lambda^{\dot\alpha j}\,\bar\eta_{rj}}{\bar M} \quad . \tag{3.23}$$

It follows from the definition (3.22) and from eqs. (3.4) that under supersymmetry the odd supertwistor variables transform as

$$\delta_\epsilon\eta_{ri} = \sqrt{2}\epsilon^\alpha_r\lambda_{\alpha i} \quad , \quad \delta_{\bar\epsilon}\bar\eta_{ri} = \sqrt{2}\bar\epsilon^{\dot\alpha}_r\bar\lambda_{\dot\alpha i}\,. \tag{3.24}$$

Finally we note that, in terms of the fermionic composite coordinates, the $\omega, \bar\omega$ components (3.20) of the two $N=2$ supertwistors can be rewritten as

$$\begin{aligned}\omega^\alpha_{\ i} &= i\Big(\bar\lambda_{\dot\beta i}\,x^{\dot\beta\alpha} + \frac{1}{\sqrt{2}}\,\lambda^{\alpha i}\,z\Big) + (\theta^\alpha_{\ r}\,\bar\eta_{ri} - \theta^{\alpha r}\eta^i_r) \quad ,\\ \bar\omega^{\dot\alpha}_{\ i} &= -i\Big(x^{\dot\alpha\beta}\,\lambda_{\beta i} + \frac{1}{\sqrt{2}}\,\bar\lambda^{\dot\alpha i}\,\bar z\Big) - (\bar\theta^{\dot\alpha}_{\ r}\,\eta_{ri} - \bar\theta^{\dot\alpha r}\,\bar\eta^i_r) \quad ,\end{aligned} \tag{3.25}$$

[4]See [21] for similar relations in the framework of D=6 Lorentz harmonics.

which are the supersymmetric generalizations of eqs. (2.4).

The supersymmetric extension (3.3) of the bosonic model (2.2) can be written in the form

$$\mathcal{L}^{\rm SUSY} = \mathcal{L}^B - i\left(\lambda_{\alpha i}\,\dot\theta^\alpha_{\ r}\,\bar\eta_{ri} + \bar\lambda_{\dot\alpha i}\,\dot{\bar\theta}^{\dot\alpha}_{\ r}\,\eta_{ri}\right) + i\left(M\theta_{\beta r}\,\dot\theta^{\beta r} + \bar M\,\bar\theta_{\dot\beta r}\,\dot{\bar\theta}^{\dot\beta r}\right)\ . \qquad (3.26)$$

A calculation now shows (modulo a total time derivative) that, after introducing (3.23), the purely supertwistorial lagrangian for our model reads

$$\begin{aligned}\mathcal{L}^{\rm SUSY} &= \mathcal{L}_1^{\rm SUSY} + \mathcal{L}_2^{\rm SUSY} \quad ,\\ \mathcal{L}_1^{\rm SUSY} &\equiv \frac{i}{2}(\bar Z^A_{\ i}\,\dot Z_{Ai} - \dot{\bar Z}^A_{\ i}\,Z_{Ai})\ ,\\ \mathcal{L}_2^{\rm SUSY} &\equiv \frac{i}{\sqrt2}\,(\eta_{ri}\,\dot{\bar\eta}_{ri} - \dot\eta_{ri}\,\bar\eta_{ri}) - \frac{i}{\sqrt2}\left(\dot\eta_{ri}\,\eta^{ri} + \dot{\bar\eta}_{ri}\,\bar\eta^{ri}\right)\ ,\end{aligned} \qquad (3.27)$$

where the scalar product of the $\bar Z^A_i = (\omega^\alpha_i\,,\,\bar\lambda_{\dot\alpha i})$ and $Z_{Ai} = (\lambda_{\alpha i}\,,\,\bar\omega^{\dot\alpha}_i)$ twistors, in which $\omega^\alpha_i\,,\,\bar\omega^{\dot\alpha}_i$ are those in (3.20), is given by eq. (2.6). Using eq. (3.23) it is seen that the $\mathcal{L}_1^{\rm SUSY}$ part of $\mathcal{L}^{\rm SUSY}$ depends only on $\eta\,,\bar\eta,\lambda\,,\bar\lambda$ and on the time derivatives of the λ's; all the dependence of $\mathcal{L}^{\rm SUSY}$ on the derivatives of the $\eta,\bar\eta$ variables is contained in $\mathcal{L}_2^{\rm SUSY}$ above.

The SU(2, 2|2)-invariant product of N=2 supertwistors (3.21) is defined by[5]

$$\bar{\mathcal{Z}}^R_{\ i}\mathcal{Z}_{Rj} \ = \bar Z^A_{\ i}\,Z_{Aj} \ + \sqrt2\,\bar\eta_{ri}\,\eta_{rj} \quad . \qquad (3.28)$$

The bosonic subgroup of SU(2, 2|2) is SU(2, 2) × U(2) ≈ $\widetilde{SO}(2,4)$ × U(2), of which SU(2, 2) acts on the A indices and U(2) on the index r; each factor preserves the two terms in (3.28) independently. Using (3.28), the lagrangian (3.27) takes the form

$$\mathcal{L}^{\rm SUSY} = \frac{i}{2}\left(\bar{\mathcal{Z}}^R_{\ i}\,\dot{\mathcal{Z}}_{Ri} - \dot{\bar{\mathcal{Z}}}^R_i\,\mathcal{Z}_{Ri}\right) - \frac{i}{\sqrt2}\left(\dot\eta_{ri}\,\eta^{ri} + \dot{\bar\eta}_{ri}\,\bar\eta^{ri}\right)\ . \qquad (3.29)$$

The first term in (3.29) is the free lagrangian for two supertwistors, which are coupled only through the second term. This last one is the pull-back to the worldline of the supertwistor space one-form $d\eta\ \eta + d\bar\eta\ \bar\eta$, a potential one-form of the closed, supersymmetry-invariant two-form $d\eta\ d\eta + d\bar\eta\ d\bar\eta$ and, thus, $\mathcal{L}_2^{\rm SUSY}$ is the supertwistorial WZ part of $\mathcal{L}^{\rm SUSY}$ (for the geometry of WZ terms, see [22]). In fact, using $\delta_\epsilon\eta$, $\delta_{\bar\epsilon}\bar\eta$ in (3.24), it is seen that $d\eta\eta + d\bar\eta\bar\eta$ is invariant modulo an exact term.

We calculate now, using (3.20), (3.22) and (2.6) the value of the scalar products (3.28), and obtain

$$\bar{\mathcal{Z}}^R_i\ \mathcal{Z}_{Rj} = \frac{i}{2}\left(Mz - \bar M\bar z\right)\delta_{ij} - \frac{1}{\sqrt2}\left(\eta_{ri}\,\eta^{rj} + \bar\eta_{ri}\,\bar\eta^{rj}\right)\ , \qquad (3.30)$$

(*cf.* (2.8)). Since $\eta_{ri}\,\eta^{ri} \equiv 0$ due to the η's odd Grassmann parity, one obtains

$$\bar{\mathcal{Z}}^R_k\ \mathcal{Z}_{Rk} = i(Mz - \bar M\bar z) \qquad (3.31)$$

[5] The different components of the (super)twistors are not dimensionally homogeneous. In natural units, $[\lambda] = L^{-\frac12}$, $[\bar\omega] = L^{\frac12}$, $[\eta] = L^0$; the scalar products of (super)twistors (eqs. (2.6), (3.28)) are, of course, dimensionless. Note also that (1.7) implies that the components λ, $\bar\omega$ and η of the (super)twistors transform in the same manner under a symmetry group acting on the index i that labels the two (super)twistors.

JHEP01(2009)041

as in the non-supersymmetric case. Proceeding as in section 2 and using (3.31), the constraints (3.30) may be rewritten just in terms of supertwistorial variables as

$$\bar{\mathcal{Z}}_i^R \, \mathcal{Z}_{Rj} - \frac{1}{2}\delta_{ij} \, \bar{\mathcal{Z}}_k^R \, \mathcal{Z}_{Rk} + \frac{1}{\sqrt{2}}\left(\eta_{ri} \, \eta^{rj} + \bar{\eta}_{ri} \, \bar{\eta}^{rj}\right) = 0 \,, \tag{3.32}$$

which extend those in (2.9) to the supertwistorial case. The constraints (3.32) and the relations (1.4) that characterize the model (3.29) can be incorporated to it by means of suitable lagrange multipliers.

We now turn to the fermionic gauge symmetries of our model.

4. κ-symmetry and κ-invariant formulation of the fermionic sector

Let us consider now the L_2 part of the action (3.27) involving the derivatives of the fermionic variables. Introducing new complex Grassmann variables ($\eta^{ri} = \epsilon^{rs}\epsilon^{ij}\eta_{sj}$ etc.)

$$\xi_{ri} \equiv \eta_{ri} + \bar{\eta}^{ri} \qquad (\,\bar{\xi}_{ri} \equiv \bar{\eta}_{ri} + \eta^{ri}\,) \quad , \tag{4.1}$$

and using the Grassmann nature of η_{ri}, $\bar{\eta}_{ri}$ one gets, up to a total derivative,

$$\mathcal{L}_2^{\mathrm{SUSY}} = \frac{i}{\sqrt{2}}\xi^{ri}\dot{\xi}_{ri} \equiv \frac{i}{\sqrt{2}}\epsilon^{rs}\epsilon^{ij}\xi_{sj}\dot{\xi}_{ri} \quad . \tag{4.2}$$

If we observe that the variables (4.1) satisfy the SU(2)-reality condition representing quaternionic structure

$$\xi_{ri} = \bar{\xi}^{ri} = \epsilon^{rs}\epsilon^{ij}\bar{\xi}_{sj} \quad , \tag{4.3}$$

we see that the action (4.2) can be written in two different ways,

$$\mathcal{L}_2^{\mathrm{SUSY}} = \frac{i}{\sqrt{2}}\bar{\xi}_{ri}\dot{\xi}_{ri} = \sqrt{2}\bar{\xi}_i\dot{\xi}_i \quad . \tag{4.4}$$

where we have chosen $\xi_i \equiv \xi_{1i}$ and used (4.3).

Thus, the fermionic action (4.4) effectively depends on two complex Grassmann variables only. The κ-transformations with Grassmann parameters ρ_{ri} that satisfy the condition (*cf.* (4.3)) such that $\bar{\rho}^{ri} = -\rho_{ri}$

$$\delta\eta_{ri} = \rho_{ri}\,, \quad \delta\bar{\eta}^{ri} = -\rho_{ri} \quad , \tag{4.5}$$

leave invariant the variables ξ_{ri} in (4.1) as well as the action (3.27). Using in (3.17) the spinor bilinears (1.2) for $p_{\alpha\dot{\beta}}$, and comparing eq. (3.17) with (4.5) one obtains, using eqs. (1.2), (2.7) and (3.22),

$$\rho_{ri} = -\bar{M}\epsilon_{rs}\lambda_i^{\alpha}\kappa_{\alpha s} + \epsilon_{is}M\bar{\lambda}_{\dot{\beta}s}\bar{\kappa}_r^{\dot{\beta}} = -\bar{\rho}^{ri} \quad . \tag{4.6}$$

It is easy to deduce the reality condition (4.3) if we assume that the fermionic Grassmann sector of the $D = 4$ supertwistor degrees of freedom is described by a single quaternionic $D = 6$ supertwistor coordinate: its Grassmann sector is given by the odd quaternionic variable $\xi = \xi_{(0)} + \xi_{(r)}e^r$, where $\xi_{(0)}$, $\xi_{(r)}$ $(r = 1, 2, 3)$ are four real Grassmann variables and $e^r e^s = -\delta^{rs} + \epsilon^{rst} e^t$. In the matrix representation obtained by replacing the quaternionic units by the Pauli matrices,

$$1 \to \sigma^0 \quad , \quad e^r \leftrightarrow -i\sigma^r \quad , \tag{4.7}$$

the quaternionic variable ξ becomes the 2×2 matrix ($\mu = 0, r$)

$$\xi = \sigma^0 \xi_{(0)} - i\sigma^i \xi_{(i)} = \begin{pmatrix} \xi_{(0)} - i\xi_{(3)} & -i\xi_{(1)} - \xi_{(2)} \\ -i\xi_{(1)} + \xi_{(2)} & \xi_{(0)} + i\xi_{(3)} \end{pmatrix} \quad . \tag{4.8}$$

It is trivial to check that ξ_{ri} as given by the elements of the matrix (4.8) satisfies the subsidiary condition (4.3); clearly, the hermitian matrix $\xi^\dagger$ describes the conjugate quaternion $\xi = \xi_{(0)} - \xi_{(r)} e^r$. We see therefore that our supertwistor model, described by the lagrangian (3.27), reflects the quaternionic structure inherent to the $D = 6$ geometry. The κ-transformations in our formulation with two independent complex $D = 4$ supertwistors represent the redundant degrees of freedom which disappear if we pass to the $N = 1$, $D = 6$ quaternionic supertwistor coordinates.

The complex Grassmann coordinates ξ_r ($= \xi_{1r}$, $r = 1, 2$) satisfy, when the canonical quantization of the action (4.4) is performed, the relations ($\hbar = 1$)

$$\{\xi_r, \bar{\xi}_s\} = \delta_{rs} \, , \quad \{\xi_r, \xi_s\} = \{\bar{\xi}_r, \bar{\xi}_s\} = 0 \quad . \tag{4.9}$$

If we supplement the above anticommutators with the canonical twistorial equal-time commutators,

$$[\lambda_{\alpha i}, \omega_j^\beta] = i\delta_\alpha^{\ \beta} \, \delta_{ij} \quad , \qquad [\bar{\lambda}_{\dot{\alpha} i}, \bar{\omega}_j^{\dot{\beta}}] = i\delta_{\dot{\alpha}}^{\ \dot{\beta}} \, \delta_{ij} \quad , \tag{4.10}$$

which follow from the symplectic twistorial two-form, we can postulate the following formulae for the four complex supercharges describing the algebraic basis of $N = 2$, $D = 4$ supersymmetry algebra

$$\begin{aligned} Q_\alpha^{(1)} &= \lambda_\alpha^{\ i} \xi_i \, , \ \bar{Q}_{\dot{\alpha}}^{(1)} = \bar{\lambda}_{\dot{\alpha}}^{\ i} \bar{\xi}_i \, , \\ Q_\alpha^{(2)} &= \lambda_{\alpha i} \bar{\xi}_i \, , \ \bar{Q}_{\dot{\alpha}}^{(2)} = \bar{\lambda}_{\dot{\alpha} i} \xi_i \quad . \end{aligned} \tag{4.11}$$

Indeed, using the canonical commutation relations (4.9), (4.10) one obtains

$$\begin{aligned} \left\{ Q_\alpha^{(r)}, \bar{Q}_{\dot{\beta}}^{(s)} \right\} &= \delta_{rs} \, \lambda_{\alpha i} \, \bar{\lambda}_{\dot{\beta} i} = \delta_{rs} \, P_{\alpha\dot{\beta}} \, , \\ \left\{ Q_\alpha^{(r)}, Q_\beta^{(s)} \right\} &= \epsilon_{rs} \, \lambda_{\alpha i} \, \lambda_\beta^{\ i} = -\frac{1}{\sqrt{2}} \, \epsilon_{rs} \, \epsilon_{\alpha\beta} M \, , \\ \left\{ \bar{Q}_{\dot{\alpha}}^{(r)}, \bar{Q}_{\dot{\beta}}^{(s)} \right\} &= \epsilon_{rs} \, \bar{\lambda}_{\dot{\alpha} i} \, \bar{\lambda}_{\dot{\beta}}^{\ i} = -\frac{1}{\sqrt{2}} \, \epsilon_{rs} \, \epsilon_{\dot{\alpha}\dot{\beta}} \, \bar{M} \, , \end{aligned} \tag{4.12}$$

which reproduces the fermionic sector of the $N = 2$, $D = 4$ supersymmetry algebra with a composite central charge M.

5. The massive $D = 4, N = 2$ superparticle model with WZ term from the $D = 6$ supertwistorial framework

Our lagrangian (3.3) may be written in terms of $D = 6$ four-component Weyl spinors. We introduce

$$\Lambda_1^A = \begin{pmatrix} \lambda_1^\alpha \\ \bar{\lambda}_{\dot{\alpha} 2} \end{pmatrix} \qquad , \qquad \Lambda_2^A = \begin{pmatrix} -\lambda_2^\alpha \\ \bar{\lambda}_{\dot{\alpha} 1} \end{pmatrix} \quad . \tag{5.1}$$

The complex spinors (5.1) satisfy the D=6 symplectic Majorana reality condition

$$\Lambda^A_r = \epsilon^{rs} C^{A\dot{A}} \Lambda^{\dot{A}}_s \ , \tag{5.2}$$

where $\Lambda^{\dot{A}} \equiv (\Lambda^A)^*$ and

$$C = \begin{pmatrix} 0 & \epsilon^{\alpha\beta} \\ -\epsilon_{\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix} , \qquad C^T = -C \, , \qquad C^2 = -1 \ , \tag{5.3}$$

is the charge conjugation matrix.

The $D = 6$ generalization of Pauli matrices can be obtained by replacing in the expression of the $D = 4$ Pauli matrices the imaginary unit by the three quaternionic imaginary units e^i $(i = 1, 2, 3)$ as follows

$$\sigma^k = \left(1_2 \, , \begin{pmatrix} 0 & -e^i \\ e^i & 0 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \quad , \tag{5.4}$$

$k = 0, i, 4, 5$. Making a similarity transformation $\sigma'^k = A \sigma^k A^{-1}$ with $A = \frac{1}{\sqrt{2}}(\sigma^2 + \sigma^3)$ $(A^2 = 1, A = A^{-1})$ and using the realization (4.7) in the expression of the σ'^k, we obtain six complex 4×4 matrices $\Sigma_{A\dot{B}}$ in the form

$$\left(1_4, \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} , - \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} , i \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix} \right) \ . \tag{5.5}$$

Since $C\Sigma C^{-1} = \Sigma^*$, the undotted A and the dotted $\dot{A}$ indices transform similarly under $SO(1,5)$ and, unlike in the $D = 4$ case, there is no metric allowing us to raise and lower the D=6 Weyl spinorial indices.

Let $\Lambda^{1A} \equiv \Lambda^A$. It follows that[6]

$$\begin{aligned}
\Lambda^A \, (\Sigma^\mu)_{A\dot{B}} \, \Lambda^{\dot{B}} &= \lambda_{\alpha i} \, (\sigma^\mu)^{\alpha\dot{\beta}} \bar{\lambda}_{\dot{\beta} i} \ = \ \sqrt{2} p^\mu \, , \\
\Lambda^A \, (\Sigma^4)_{A\dot{B}} \, \Lambda^{\dot{B}} &= -\frac{1}{\sqrt{2}}(M + \bar{M}) = \sqrt{2} p^4 \, , \\
\Lambda^A \, (\Sigma^5)_{A\dot{B}} \, \Lambda^{\dot{B}} &= -\frac{i}{\sqrt{2}}(M - \bar{M}) = \sqrt{2} p^5 \, ,
\end{aligned} \tag{5.6}$$

where we have used eqs. (1.2) ((1.4)) in the first (second and third) expression above and that $M = p_4 - ip_5 \, , \ \bar{M} = p_4 + ip_5$. This gives the algebraic D=6 zero mass shell condition,

$$p_k p^k = p_\mu p^\mu - p_4^2 - p_5^2 = 0 \ . \tag{5.7}$$

Then, the bosonic lagrangian (2.2) may be now written in a six-dimensional form as the lagrangian for the D=6 massless particle in a hybrid spinorial/spacetime formulation,

$$\mathcal{L}^B = \sqrt{2} \Lambda^A \, \dot{x}_{A\dot{B}} \, \Lambda^{\dot{B}} \ , \tag{5.8}$$

[6]Since the Σ's are now four-dimensional and $\tilde{\Sigma} = \Sigma^{\dot{B}A} \equiv (\Sigma^0, -\Sigma^s)$, $s = 1, \ldots, 5$, $\Sigma^k \widetilde{\Sigma}^l + \Sigma^l \widetilde{\Sigma}^k = 2\eta^{lk} 1_4$ $(\eta_{kl} = \mathrm{diag}(1, -1, \ldots -1))$ we define now $a_{A\dot{B}} \equiv \frac{1}{2} \Sigma^k_{A\dot{B}} a_k$, $b^{\dot{B}A} \equiv \frac{1}{2} \Sigma^{\dot{B}A\ k} b_k$. As a result, we have $a_{A\dot{B}} b^{\dot{B}A} = a \cdot b$ as for D=4, but now $\frac{1}{2} a_{A\dot{B}} \Sigma^{\dot{B}A\ k} = a^k$, $\frac{1}{2} a^{\dot{B}A} \Sigma^k_{A\dot{B}} = a^k$, $k = \mu, 4, 5$. The various $\sqrt{2}$'s in (5.6) come from having D=4-adapted factors in eqs. (1.2), (1.4) (see footnote 1) within a D=6 context.

JHEP01(2009)041

where

$$\dot{x}_{A\dot{B}} = \frac{1}{2}\dot{x}_k(\Sigma^k)_{A\dot{B}} \,, \tag{5.9}$$

and the D=6 zero mass condition (5.7) is built in algebraically.

To supersymmetrize the bosonic lagrangian (5.8) we introduce D=6 superspace with Weyl-Grassmann spinors. The lagrangian (3.3) is then obtained by the replacement

$$\dot{x}^{\dot{B}A} \longrightarrow \omega^{\dot{B}A} = \dot{x}^{\dot{B}A} - 2i(\dot{\theta}^A\,\bar{\theta}^{\dot{B}} - \theta^A\,\dot{\bar{\theta}}^{\dot{B}})\,, \tag{5.10}$$

where we use the following four-component D=6 Grassmann spinors

$$\theta^A = \begin{pmatrix} \theta^\alpha_{\ 1} \\ \bar{\theta}_{\dot{\alpha}2} \end{pmatrix}\,, \qquad \theta^{\dot{A}} = \begin{pmatrix} \bar{\theta}^{\dot{\alpha}}_{\ 1} \\ \theta_{\alpha 2} \end{pmatrix}\,. \tag{5.11}$$

The substitution (5.10) may be now written in terms of a pair of two-dimensional $D = 4$ Weyl spinors as

$$\begin{aligned} \dot{x}_\mu \longrightarrow \omega_\mu &= \dot{x}_\mu - i(\dot{\theta}^\alpha_{\ r}(\sigma_\mu)_{\alpha\beta}\,\bar{\theta}^{\dot{\beta}}_{\ r} - \theta^\alpha_{\ r}(\sigma_\mu)_{\alpha\dot{\beta}}\,\dot{\bar{\theta}}^{\dot{\beta}}_{\ r})\,, \\ \dot{z} \longrightarrow \omega &= \dot{z} + 2i\,\theta_{\alpha r}\,\dot{\theta}^{\alpha r}\,, \\ \dot{\bar{z}} \longrightarrow \bar{\omega} &= \dot{\bar{z}} + 2i\,\bar{\theta}_{\dot{\alpha}r}\dot{\bar{\theta}}^{\dot{\alpha}r}\,. \end{aligned} \tag{5.12}$$

It may be checked that the supersymmetric lagrangian (3.3) in $D = 6$ notation can be written as follows

$$\mathcal{L}^{\text{SUSY}} = \sqrt{2}\Lambda^A\,\omega_{A\dot{B}}\,\Lambda^{\dot{B}} = \frac{1}{\sqrt{2}}\omega_k\Lambda^A\Sigma^k_{A\dot{B}}\Lambda^{\dot{B}} \quad , \tag{5.13}$$

where $k = 0, 1, \ldots 4, 5$ and $\omega^k = (\,\omega^\mu\,,\,\frac{1}{2}(\bar{\omega}+\omega)\,,\,\frac{i}{2}(\bar{\omega}-\omega)\,)$ and ω and $\bar{\omega}$ are given in eq. (3.2).

In our model (see (3.3) or (5.13)) the central charge coordinates z, $\bar{z}$, as well as the dual central charges M, $\bar{M}$, are dynamical variables; however the central charges are constants on-shell (the field equations are $\dot{M} = \dot{\bar{M}} = 0$). The static approximation $M = \bar{M} = const.$ can be achieved consistently in our first order formulation by the $D = 6 \to D = 4$ reduction procedure in target space. We set in eq. (5.13))[7]

$$p_4 = p_5 = \text{const.} \qquad\qquad p_4^2 + p_5^2 = m^2 = \text{const.} \tag{5.14}$$

Putting

$$p_4 = m\sin\varphi\,, \qquad\qquad p_5 = m\cos\varphi\,, \qquad\qquad M = me^{-i\varphi}\,, \tag{5.15}$$

where φ is a constant phase, the dimensional reduction $\mathcal{L}^{\text{SUSY}} \to \mathcal{L}^{\text{SUSY}}_{D=4}$ gives

$$\mathcal{L}^{\text{SUSY}}_{D=4} = p_\mu\omega^\mu + im\left(e^{-i\varphi}\,\theta_{\alpha r}\,\dot{\theta}^{\alpha r} + e^{i\varphi}\,\bar{\theta}_{\alpha r}\,\dot{\bar{\theta}}^{\alpha r}\right) + \frac{e}{2}(p^2 - m^2)\,, \tag{5.16}$$

[7] See [23, 24]; for the application of the dimensional reduction procedure to superparticles see [25]–[27]. There, one performs the dimensional reduction procedure in target space, in consistency with the on-shell values of the reduced solutions. Note that there is no need of restricting x_4, x_5 because in the first order formalism the only term in the lagrangian depending on these variables becomes a total derivative for constant p_4, p_5.

where the D=4 mass shell condition, eq. (5.7) after using (5.15), is imposed by a Lagrange multiplier. We further observe that we can set $\varphi = 0$ because ω^μ is invariant under the constant phase transformations

$$\theta'_{\alpha r} = e^{\frac{i}{2}\varphi}\,\theta_{\alpha r} \quad , \quad \bar{\theta}'_{\dot{\alpha} r} = e^{-\frac{i}{2}\varphi}\,\bar{\theta}_{\dot{\alpha} r} \quad . \tag{5.17}$$

Subsequently, we obtain the first order formulation of the $D = 4$, $N = 2$ superparticle model with WZ term introduced by the two of present authors [18]. Indeed, after eliminating p_μ and e from (5.16) by the algebraic field equations we obtain

$$\mathcal{L}^{\text{SUSY}}_{D=4} = m\sqrt{\omega_\mu\,\omega^\mu} + i\,m(\theta_{\alpha r}\,\dot{\theta}^{\alpha r} + \bar{\theta}_{\alpha r}\,\dot{\bar{\theta}}^{\alpha r})\,. \tag{5.18}$$

The model (5.18) corresponds to the case where the central charge is represented by a constant real mass parameter. It is worth stressing here that the equality of parameters m in front of the first (Nambu-Goto-like) and second (WZ) term in the Lagrangian (5.18) corresponds in our two-supertwistor model to the equality of the numerical coefficients in front of the two terms in (3.27), necessary for the invariance under the local κ-transformations (4.5) in two-supertwistor space. In the $N = 2$ superspace formulation, the equality of the 'bosonic' and 'fermionic' masses in the two terms of the lagrangian (5.18) allows as well for the invariance under the κ-gauge transformations (3.16), (3.19), which are necessary to balance the number of fermionic and bosonic degrees of freedom in the p=0 super-p-brane model, as it is the case for extended objects in general [28].

6. Discussion

We have considered here the supertwistorial formulation of D=4 superparticles with mass and $N = 2$ supersymmetry. Our interest in the massive case is due to the fact that it is the massive superparticle model with WZ term [18], rather than the massless one, which is the pointlike p=0 analogue of the extended $p > 0$ super-p-branes. By constructing a model with two $N = 2$, $D = 4$ supertwistors we have been able to study the appearance of both the WZ term and the fermionic κ-transformations in a (super)twistorial framework.

It is known that massless superparticles with N-extended supersymmetry can be described by a single N-extended supertwistor $\mathcal{Z} = (\lambda_\alpha, \bar{\omega}^{\dot{\alpha}}, \eta_r)$ with N complex Grassmann coordinates η_r, $r = 1 \ldots N$. The degrees of freedom of one superstwistor are invariant under κ-transformations *i.e.*, for massless superparticles the coordinates of the single supertwistor already describe the 'physical' degrees of freedom. Thus, since the N-extended $D = 4$ superspace contains $2N$ complex Grassmann coordinates $\theta_{\alpha r}$ ($\alpha = 1, 2$, $r = 1, \ldots N$), the equivalence between the supertwistorial and the superspace formulations of the massless superparticle requires the removal of half of the odd superspace degrees of freedom by means of the fermionic gauge κ-transformations [18, 19].

However, if we wish to describe a D=4 massive superparticle in a supertwistorial approach, we necessarily need at least two supertwistors to allow for a non-vanishing mass [7, 8, 3, 9]–[13]. In this paper we have considered the $N = 2$ supersymmetry case using for two $D = 4$ supertwistors, which give rise to a supertwistorial WZ term and to the

κ-gauge transformations. Indeed, it turns out that the number of Grassmannian degrees of freedom of our supertwistorial model is the same as in N=2 superspace (see eqs. (3.23)), and thus the familiar local fermionic transformations of the superspace framework must appear as well in the purely two-supertwistorial description.

In order to obtain the κ-invariant formulation of our model we observe that two $N = 2$, $D = 4$ supertwistors can be obtained from a single $N = 1$, $D = 6$ supertwistor provided that the odd $D = 4$ supertwistor coordinates satisfy the SU(2)-Majorana reality condition (eq. (4.3)). One can conclude therefore that our κ-transformations account for the degrees of freedom that disappear when we use such a pair of constrained $D = 4$ complex supertwistors, equivalent to the single $D = 6$ supertwistor with associated quaternionic geometry.

For the D=4, N-extended supersymmetry case one can introduce $\frac{N(N-1)}{2}$ mass-like parameters corresponding to as many complex central charges ($i = 1 \ldots N > 2$) [29]

$$\begin{gathered} \left\{ Q^i_\alpha, \bar{Q}^j_{\dot\beta} \right\} = \delta^{ij}\, P_{\alpha\dot\beta} \quad , \\ \left\{ Q^i_\alpha, Q^j_\beta \right\} = \epsilon_{\alpha\beta}\, \Xi^{ij} \quad , \quad \left\{ \bar{Q}^i_{\dot\alpha}, \bar{Q}^j_{\dot\beta} \right\} = \epsilon_{\dot\alpha\dot\beta}\, \bar{\Xi}^{ij} \quad , \end{gathered} \tag{6.1}$$

where $\Xi^{ij} = -\Xi^{ji}$ is the complex $N \times N$ skewsymmetric matrix of generators of the central charges . To introduce in a supertwistorial formalism all possible massive parameters as independent spinorial bilinears, one can generalize the relation $\Xi = M$ (eq. (1.6)) to allow for antisymmetric charges as follows

$$\Xi^{ij} = -\Xi^{ji} \propto \lambda^i_\alpha\, \lambda^{\alpha j} \,, \quad i, j = 1, \ldots N \tag{6.2}$$

(*cf.* (1.4)). For such a purpose N independent copies of N-extended supertwistors (1.7) are needed, with N^2 complex fermionic degrees of freedom[8]. Superparticle models characterized by having several mass-like parameters corresponding to the central charges (6.2) are not known, but by generalizing of our two-supertwistor framework one may guess how to construct a supertwistor lagrangian in terms of $N > 2$ copies of N-extended supertwistors. One of the primary tasks in building such a model in D=4 would be to describe the corresponding generalized κ-transformations which would require the maximal number $N(N-1)$ of odd complex parameters.

The most interesting case one could study is that of a $D = 4$, $N = 4$ model with 12 κ-gauge odd parameters. If we could introduce four $D = 4$ supertwistors with suitable constraints to describe the degrees of freedom of an octonionic $N = 1$, $D = 10$ supertwistor [6], the κ-invariant formulation would determine the corresponding $N = 4, D = 4$ supertwistor dynamics with octonionic structure[9].

We conclude by mentioning that, recently, there has been a renewed interest in twistor theory and in the general Penrose programme as a result of the applications of twistors

[8]Since for D=4 there are two linearly independent constant spinors, the $N > 2$ case is useful for $D > 4$ (the number of components of a Dirac spinor grows as $2^{[\frac{D}{2}]}$).

[9]In such a formalism a single octonion coordinates spanning $\mathbb{R}^8$ would be described by four complex split octonionic units (see e.g. [36]). It is unclear, however, whether the problems associated with non-associativity can be avoided.

and supertwistors in various modern contexts as e.g., in the analysis and computation of $N=4$ Yang-Mills amplitudes [30, 31], in various (super)string models [30, 32−34] or in connection with an algebraic description of the BPS states in M-theory [35]. One may assume, therefore, that the study of dynamical multi-supertwistorial models is a useful step towards a further application of (super)twistorial ideas to fundamental interactions formalisms.

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JHEP01(2009)041